Description of Heavy Quark Systems by means of Energy Dependent Potentials

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We apply, for the first time, an energy dependent Schrödinger equation to describe static properties of heavy quark systems, i.e. charmonium and bottonium. We show that a good description of the eigenstates and reasonable values for the widths can be obtained. Values of the radii and of the density at the origin are also given. We compare the results to those deduced with a Schrödinger equation implemented with potentials used so far. We note that the energy dependence of the confining potential provides a natural mechanism for the saturation of the spectra. Our results introduce a new class of potentials for the description of heavy quark systems.

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Wave equations with energy dependent potentials are already familiar in physics. The Pauli-Schrödinger equation describing a particle in an external electromagnetic field [1] or the Hamiltonian formulation of relativistic quantum mechanics, in connection with manifestly covariant formalism with constraints [2, 3, 4], offer two examples of energy dependent potentials. Such dependence arises from the relativistic nature of the problem. Moreover, it has been shown that in the context of relativistic quantum mechanics, the potentials can be derived from fundamental theories, i.e a link exists with field theory [5]. The present work has a different aim, namely to show the characteristic features which can be expected from energy dependent potentials, when applied to a specific physical system. For this purpose, simple potentials admitting analytic or semi-analytic solutions are of great help since they facilitate the comparison with usual potentials.

The introduction of an energy dependence has several implications with respect to usual quantum mechanics. For example, the conservation of the norm asks for the modification of the scalar product [6]. The energy dependent potentials must fulfill certain conditions in order to result in a meaningful quantum theory [7]. For those potentials which are acceptable, the corresponding Schrödinger equation is equivalent to a Schrödinger equation including a non-local potential. The non-locality, however, is treated more efficiently under the form of an energy dependent potential. A detailed analysis of these formal aspects is performed in [7] showing that the properties of a good quantum theory are well preserved.

The scope of this letter is to apply, for the first time, such an approach to cases of physical interests. A natural application is offered by heavy quark systems. Their description has been a quite successful playground for potential models in the past [8, 9, 10, 11, 12, 13]. Here we show that energy dependent potentials represent a new class of potentials that can account for the prop-

erties of charmonium and bottonium. In particular, we present the eigenstates of these systems, the root mean square radii, the leptonic widths and the density of the S-wave functions at the origin, the latter being a useful input for quarkonium production in high-energy colisions. We compare our results to those obtained with the potentials known so far to give reasonable description of charmonium and bottonium properties. We take the Buchmüller and Tye (BT) [8] and the Cornell [9] potentials as examples.

For simplicity we consider spherical symmetric potentials, assuming a linear energy dependence. The corresponding Schrödinger equation reads (ignoring spin degrees of freedom):

$$[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(r,E)]\Psi_{n,\ell}(\vec{r}) = E_{n,\ell}\Psi_{n,\ell}(\vec{r}), \qquad (1)$$

where the potential used is:

$$V(r,E) = (1 + \gamma E)f(r) + D_0\vec{\ell}^2 + V_0. \qquad (2)$$

with n, ℓ denoting the principal and the orbital quantum numbers respectively and $m/2$ the reduced mass (we work in units $\hbar$=197.3 MeV fm^{-1}). To illustrate our aim we choose two radial dependences for which analytical solutions exist: I) the linear one, i.e. $f(r) = \lambda m r$ and II) the harmonic one, i.e. $f(r) = m\omega^2 r^2/2$. In each case the total number of parameters is 5. Note that only four of the parameters are free since $M(q\bar{q}) = 2m + V_0$. They are determined by constraining the energy differences between the ground state and the 1P, 2S, 2P and 3S states to the experimental energies for both systems. The 1D state is also considered for the $c\bar{c}$ system. Note that for the two chosen radial dependence, only negative values of γ are acceptable [7].

The term $D_0\vec{\ell}^2$ is added to break the degeneracy between the states of different angular momentum. Its contribution is necessary to achieve accurate fits, especially in the case of the harmonic oscillator.

In the harmonic dependence case, the energies are ob-

tained by solving the following quadratic equation:

$$E_{n,\ell}^2 \;-\; \left[2D\ell(\ell+1) + a\hbar^2\omega^2\gamma\right]E_{n,\ell} \tag{3}$$
$$+\; D^2\ell^2(\ell+1)^2 - a\hbar^2\omega^2 = 0$$

where $D = \frac{\hbar^2}{2m}D_0$ and $a = (4n+2\ell+3)^2/4$; while for the linear dependence the S-state eigenvalues are obtained by solving

$$(E_{n,0} + V_0)^{3/2} - A_n\sqrt{m}\lambda(1 + \gamma E_{n,0}) = 0 \;, \tag{4}$$

with $A_n = (-a_n)^{3/2}\frac{\hbar}{\sqrt{2}}$, a_n being the n-th zero of the Airy function.

Eqs. (3) and (4) are non-linear. Therefore one has to specify the choice of the eigenvalue in the case when several solutions exist. For the harmonic oscillator, only positive energies lead to square integrable wave functions (see [7]). Consequently negative solutions are rejected. For the linear potential, Eq. (4) can be transformed into a cubic equation, which has a single real root in the range of parameters used in the present work.

In the framework of potential models, the leptonic widths of the S-states, without radiative and relativistic corrections [14, 15, 16], are given by the van Royen-Weisskopf formula [17]:

$$\Gamma_{ee}(nS) = \frac{16\pi e_q^2\alpha^2}{M^2(q\bar{q})}|R_{n,0}(0)|^2 \tag{5}$$

where $R_{n,0}$ indicates the radial component of the wave function evaluated at the origin, e_q the quark charge and $M(q\bar{q})$ is the physical mass. We have checked that the same formula holds when using energy dependent potentials.

Table I shows the sensitivity of the results to the variation of the parameters, in the case of the harmonic oscillator radial dependence as an example. The remarkable fact is the change in the spectrum pattern brought by the energy dependence, which deepens the ground state with respect to the other states. The usual harmonic oscillator would be unable to produce the relative positions of the 1P, 2S, 1D and 3S levels, even in the presence of the $\vec{\ell}^2$ term in the Hamiltonian. On the other hand, the energy dependent potential with the harmonic oscillator radial shape is not able to account for both the 1D and 3S states simultaneously. In this respect, the linear potential comes out to give much better description of the $c\bar{c}$ and $b\bar{b}$ properties.

In Tables II and III we give the results, obtained with the linear dependence, for the energies, the leptonic widths, the root mean square radius, and the density of the 1S states at the origin. We compare our results with those obtained with previous potentials, in particular the BT and the Cornell ones. Their values of the 1S energy density at the origin are taken from [12]. We see that a very good agreement on the spectrum is obtained. On the other hand the width of the 1S state as well as the

SPECTRUM					EXP.
1P	397	397	397	397	397
2S	589	589	589	589	589
1D	758.5	759.2	706.2	706.4	710
2P	839.0	839.6	789.5	789.6	840
3S	946.2	947.1	876.7	876.8	940
W.F. PROPERTIES					
Γ_{1S}	2.6	6.2	2.9	7.7	5.26(.37)
Γ_{1S}/Γ_{2S}	1.7	1.5	2.3	1.7	2.4
$\langle r^2\rangle_{1S}^{1/2}$	0.490	0.374	0.469	0.346	
PARAMETERS					
ω (fm^{-1})	2.7	2.3	3.5	3.0	
V_0 (MeV)	0.0	600.	0.0	600.	
D_0 (MeV)	2.8	4.3	2.4	3.6	
$\gamma \cdot 10^{-4}$ (MeV^{-1})	-4.29	-5.72	-4.75	-6.64	
m(GeV)	1.212	1.812	1.138	1.738	

TABLE I: SENSITIVITY OF THE $c\bar{c}$ PROPERTIES TO VARIATIONS OF THE PARAMETERS: The results display the energy differences (MeV) with respect to the ground state, the leptonic widths (keV), the root mean square radius (fm) in the case of the harmonic oscillator radial dependence. Experimental data are given for comparison, when available.

	This work	BT	CORNELL	EXP.
1P	397	424	428	397
2S	589	601	590	589
1D	710	716	713	710
2P	840	872	871	840
3S	943	1023	1015	940
Γ_{1S}	7.51	8.0	14.33	5.26(.37)
Γ_{1S}/Γ_{2S}	2.18	2.17	2.22	2.4
$\langle r^2\rangle_{1S}^{1/2}$	0.33	0.42	0.47	
$|R_{1S}(0)|^2$	0.761	0.810	1.454	

TABLE II: $c\bar{c}$ PROPERTIES: Results obtained with an energy dependent potential and the linear radial dependence, in comparison with those of Buchmüller and Tye (BT) [8] and of the Cornell [9] potentials. The values listed correspond to the energies (MeV) with respect to the ground state, to the leptonic widths (keV), the root mean square radius (fm), the density at the origin (GeV^3). The parameters used are λ=2 fm^{-1},V_0=1940 MeV, D_0=2 MeV, γ=-5.007 10^{-4} MeV^{-1} and m= 1913 MeV. Experimental data are also given.

ratio of the 2S to the 1S widths present the same quantitative agreement with experiment as previous potentials.

In this respect, one should remember that the van Royen-Weisskopf formula should be multiplied by a fac-

	This work	BT	Cornell	exp.
1P	426	430	498	428
2S	563	560	591	563
1D	798	680	747	
2P	818	790	852	792
3S	895	890	936	895
4S	1116	1160	1213	1120
Γ_{1S}	0.93	1.68	3.66	1.32(.05)
Γ_{1S}/Γ_{2S}	2.73	2.2	2.17	2.54
$\langle r^2 \rangle_{1S}^{1/2}$	0.23	0.23	0.20	
$|R_{1S}(0)|^2$	3.65	6.477	14.08	

TABLE III: $b\bar{b}$ PROPERTIES: Same as Table II, but here the results correspond to λ=0.6 fm^{-1}, V_0=2725 MeV, D_0=15 MeV, γ=-1.06 10^{-3} MeV^{-1} and m= 5475 MeV.

tor $(1-16\alpha_s/3\pi)$ corresponding to the (first-order) radiative corrections [8, 12]. In [18, 19] it is shown that after the inclusion of the relativistic corrections, potential models having $|R_{1S}(0)|^2$=0.8-1 (6-8) GeV^3 for the J/ψ (Υ) are able to reproduce the leptonic widths. This point should be reconsidered in the framework of energy dependent potentials. Here we shall merely compare the zeroth order values with the ones obtained from other potentials. In the case of the $c\bar{c}$ system, the 1S energy density at the origin is close to the value given by the BT potential. On the other hand, for the $b\bar{b}$ system, it is sensibly lower than the BT value. This situation is quite strange, since the value of $|R_{1S}(0)|^2$ is closely related to the positions of the S-states, which are very well reproduced by the present model (and is not due to a peculiar set of parameters). Indeed the density at the

origin of the 1S-state can hardly be increased by 10-15 % by choosing a lower coupling constant λ. Consequently, such a behaviour may point to the necessity of finding a more universal radial shape and/or of a more pertinent energy dependence.

In conclusion, we show that a new class of potentials, namely energy dependent potentials, give a good description of charmonium and bottonium properties. We emphasize that the energy dependence is rather naturally introduced, since it arises from the relativistic aspect of the problem. As a starting point, we have chosen potentials admitting analytical solutions, at least for the S states. This is a great advantage in view of the high nonlinearity of the differential equation to be solved. However, it could be rewarding to study other radial shapes, as well as combinations of usual and energy dependent potentials. As far as the energy dependence is concerned, the linear one has the advantage of simplicity in many respects as already emphasized in [7].

We would like to stress a very important feature of the energy dependence for confining potentials, namely the saturation of the spectra. For usual potential, $E_{n,\ell}$ increases regularly toward ∞ with n or ℓ. In contrast for energy dependent confining potentials, $E_{n,\ell}$ tends to a limit as n and ℓ increase. Thus, the density of states becomes rapidly very large, the splittings getting infinitesimally small, which forbids the observation of individual states. We believe that this property could have relevant implications in many aspects of the quark-model physics.

Finally, understanding the connection between the phenomenological potentials used here and more fundamental theories of elementary particles deserves further studies. The application to other systems is foreseen.

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